Ghost elimination in quantum gravity

1976 J. Phys. A: Math. Gen. 9 59

(http://iopscience.iop.org/0305-4470/9/1/011)

J. Phys. A: Math. Gen., Vol. 9, No. 1, 1976. Printed in Great Britain. © 1976

Ghost elimination in quantum gravity

M Nouri-Moghadam and J G Taylor
Department of Mathematics, King's College, Strand, London WC2R 2LS, UK

Received 26 June 1975

Abstract. We study the problem of quantizing the gravitational field in interaction with scalar mesons when gravity has certain non-minimal terms in addition to Einstein's Lagrangian. The extra poles in the graviton propagator are described and we attempt to remove the ghost particle in lowest-order graphs. We find that this is not possible, and exhibit a 2-meson–1-graviton scattering graph whose S matrix contribution has a ghost pole at the 2-meson threshold. We also note that if graviton ghost cancellation were possible the usual renormalizability criterion is required to prevent meson ghost appearances.

1. Introduction

Many attempts have been made recently to quantize gravity ('t Hooft and Veltman 1974, Capper and Duff 1974, Capper *et al* 1974, Deser and van Nieuwenhuizen 1974a, b, De Witt 1964, 1967, Nouri-Moghadam and Taylor 1975a, b). These stem from various reasons, from the extreme physical one of attempting to deal with matter at the end-point of collapse in a black hole to the mathematical problem of extending quantization schemes to a particular sort of gauge theory. These attempts have now been recognized as being unsuccessful (Nouri-Moghadam and Taylor 1975a, b) since the necessary counter terms to make predictions of the theory finite have either proved infinite in number or correspond to ghosts. We must still attempt to quantize gravity, however, since the reasons for considering the problem in the first place are pressing. In this paper we wish to continue the quantization scheme by taking the ghosts seriously into account and trying to find conditions on the stress–energy tensor of matter so that the ghost contributions are cancelled completely.

We start with the Einstein Lagrangian for the gravitational field coupled minimally to matter contributions, as dictated by the spin 2 character of the gravitational field. It is possible to add non-minimal contributions to the Lagrangian. However it was pointed out by Feynman (1963, 1962–3 unpublished) and others (Gupta 1968) and shown more recently by Deser (1970) by a very elegant argument, that such terms do not arise if a theory of spin 2 massless particles is allowed to have any non-trivial self-interaction generated by gauge invariance arguments from an initially free theory. This has been extended by Boulware and Deser (1974 unpublished) to prove that the long-range effects in a gauge invariant S matrix theory are those arising purely from the Einstein Lagrangian with added minimally coupled matter terms.

It has been shown that after field quantization the counter terms required to remove all single-loop ultraviolet divergences are of non-minimal character, such as R^2 or $R_{\mu\nu}R^{\mu\nu}$ in the usual notation (De Witt 1964, 1967). It is not possible to stop the analysis of the ultraviolet divergences at this point. Infinities from both higher loops and single loops containing one or more vertices generated by the counter terms must also be considered. The latter have been considered briefly elsewhere (Nouri-Moghadam and Taylor 1975a, b) and have been shown to require an ever increasing number of counter terms. This is under the assumption that the initial single-loop counter terms can be treated purely by perturbation theory.

The terms R^2 and $R_{\mu\nu}R^{\mu\nu}$ involve terms quadratic in the field variables which are of fourth order in their derivatives. The treatment of such contributions by perturbation theory would appear to be much at variance with that usual in canonical quantization. However if the total Lagrangian is treated by standard methods (Pais and Uhlenbeck 1950) it will produce ghost particles in the spin 2 field with negative energies and norms. These destroy the unitarity and positive energy spectrum of the theory.

The only way open to proceed with a quantum theory of gravity based on gauge invariance premises is to attempt to remove the ghost contributions by suitable decoupling. We will attempt to analyse that question here by looking at ghost contributions in lowest-order perturbation theory. This is itself of limited validity but can give some indication of whether any success can be achieved, and what the sort of condition might be to obtain success to all orders.

2. The graviton–ghost propagator

Let us consider the gravitational field $g_{\mu\nu}(x)$ and its associated tensors $g^{\mu\nu}$, $R_{\alpha\beta\gamma\delta}$, $R_{\mu\nu}$, R using the usual notion (De Witt 1964, 1967). We will attempt to quantize the field as described by the Lagrangian density

$$L = \sqrt{(-g)}(R+aR^2+4bR_{\mu\nu}R^{\mu\nu}). \tag{2.1}$$

We expand the field variable as

$$g_{\mu\nu} = \eta_{\mu\nu}+h_{\mu\nu}$$

where $h_{\mu\nu}$ is the field to be quantized and $\eta_{\mu\nu}$ is the Minkowski background. To first order in $h_{\mu\nu}$ we have that ('t Hooft and Veltman 1974)

$$g^{\mu\nu} = \eta_{\mu\nu}-h_{\mu\nu}$$

$$\sqrt{-g} = 1+\tfrac{1}{2}h_{\mu}{}^{\mu}$$

where indices in $h_{\mu\nu}$ are raised or lowered by $\eta_{\mu\nu}$. To the same order of approximation

$$R = \tfrac{1}{4}h^{\alpha}_{\alpha,\mu}h^{\beta,\mu}_{\beta}-\tfrac{1}{4}h^{\beta}_{\alpha,\nu}h^{\alpha,\nu}_{\beta}-\tfrac{1}{2}h^{\alpha}_{\alpha,\beta}h^{\beta,\mu}_{\mu}+\tfrac{1}{2}h^{\nu,\alpha}_{\beta}h^{\beta}_{\alpha,\nu}$$

$$R^2 = (h^{\beta,\alpha}_{\beta,\alpha}-h^{\beta,\alpha}_{\alpha,\beta})^2$$

$$R_{\mu\nu}R^{\mu\nu} = \tfrac{1}{4}(h^{\gamma}_{\gamma,\mu\nu}-h^{\gamma}_{\mu,\nu\gamma}-h^{\gamma}_{\nu,\mu\gamma}+h_{\mu\nu,\gamma}{}^{\gamma})(h^{\delta}_{\delta,\mu\nu}-h^{\delta}_{\mu,\nu\delta}-h^{\delta}_{\nu,\mu\delta}+h_{\mu\nu,\delta}{}^{\delta}).$$

To equation (2.1) we must add a symmetry breaking term which fixes the appropriate gauge; this we take to be the harmonic gauge, so we have to second order, in addition to that from equation (2.1), the symmetry breaking term

$$L^1 = -\tfrac{1}{2}(h^{\nu}_{\mu,\nu}-\tfrac{1}{2}h^{\nu}_{\nu,\mu})(h^{\lambda}_{\mu,\lambda}-\tfrac{1}{2}h^{\lambda}_{\lambda,\mu}).$$

We can thus calculate the second-order term in h in $(L+L^1)$ to be, to within terms which are total derivatives (which we can neglect), the quadratic form

$$(L+L^1) = h^{\alpha\beta}P_{(4)\alpha\beta,\mu\nu}h^{\mu\nu}$$

where

$$P_{(4)\alpha\beta,\mu\nu} = [(a+b)\Box^2+\tfrac{1}{8}\Box]\delta_{\alpha\beta}\delta_{\mu\nu}+\tfrac{1}{2}(b\Box^2-\tfrac{1}{4}\Box)(\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\beta\mu}\delta_{\alpha\nu})$$
$$+(a+2b)\partial_\alpha\partial_\beta\partial_\mu\partial_\nu-(a+b)\Box(\delta_{\alpha\beta}\partial_\mu\partial_\nu+\delta_{\mu\nu}\partial_\alpha\partial_\beta)$$
$$-\tfrac{1}{2}b\Box(\delta_{\alpha\mu}\partial_\beta\partial_\nu+\delta_{\beta\nu}\partial_\alpha\partial_\mu+\delta_{\alpha\nu}\partial_\beta\partial_\mu+\delta_{\beta\mu}\partial_\alpha\partial_\nu) \tag{2.2}$$

where $\partial_\alpha = \partial/\partial x^\alpha$ and $\Box = \partial_\alpha\partial^\alpha$. The total graviton propagator $Q^{\mu\nu,\beta\sigma}$ is thus the inverse of equation (2.2):

$$P_{(4)\alpha\beta,\mu\nu}Q^{\mu\nu,\rho\sigma} = \tfrac{1}{2}(\delta_\alpha{}^\rho\delta_\beta{}^\sigma+\delta_\alpha{}^\sigma\delta_\beta{}^\rho) = Q^{\rho\sigma,\mu\nu}P_{(4)\mu\nu,\alpha\beta}. \tag{2.3}$$

The Lagrangian (2.1) which we are quantizing has quartic derivatives in both the interaction and the free-field parts. The former presents great difficulties due to the apparent lack of an unambiguous ordering. A similar difficulty arises in the nonlinear chiral theory (Taylor 1974 and references therein), but can be removed there, at least to lowest order in perturbation theory, by careful use of the $SU(2)\times SU(2)$ symmetry of the theory. The term proportional to $\delta^4(0)$, also arising from the derivative interaction, was shown by Taylor (1974) to be cancelled to all orders (see also Keck 1971). We note that the $\delta^4(0)$ terms are automatically cancelled (Capper and Liebrandt 1973) by the dimensional regularization method we use later, though that is no true justification for their neglect.

There are other difficulties associated with the quantization of equation (2.1)†‡, but we will turn directly to the details of the Feynman propagator for equation (2.1) without discussing these further, since they add little to the problem. In order to invert $P_{(4)}$ we note that it would be possible to use the method of Rivers (1964), though since that is only for fields with a given mass we will use a more direct approach here. We represent it as a linear combination of a set of six fourth-rank tensors $T_1 \ldots T_6$ defined by

$$T_1{}^{\alpha\beta,\mu\nu} = \delta^{\alpha\beta}\delta^{\mu\nu}$$
$$T_2{}^{\alpha\beta,\mu\nu} = \tfrac{1}{2}(\delta^{\alpha\mu}\delta^{\beta\nu}+\delta^{\alpha\nu}\delta^{\beta\mu})$$
$$T_3{}^{\alpha\beta,\mu\nu} = \delta^{\alpha\beta}\partial^\mu\partial^\nu$$
$$T_4{}^{\alpha\beta,\mu\nu} = \delta^{\mu\nu}\partial^\alpha\partial^\beta$$
$$T_5{}^{\alpha\beta,\mu\nu} = \partial^\alpha\partial^\beta\partial^\mu\partial^\nu$$
$$T_6{}^{\alpha\beta,\mu\nu} = \tfrac{1}{4}(\delta^{\alpha\mu}\partial^\beta\partial^\nu+\delta^{\alpha\nu}\partial^\beta\partial^\mu+\delta^{\beta\mu}\partial^\alpha\partial^\nu+\delta^{\beta\nu}\partial^\alpha\partial^\mu).$$

† In the present situation the ordering problem is decidedly non-trivial. We have considered it briefly elsewhere though without success either for the $[\delta^3(0)]^2$ terms or the further $\delta^4(0)$ terms. Here we will not discuss these terms further but will tacitly assume that they are cancelled completely by suitable conditions. Only after we have obtained a sensible theory at the S matrix or Green function level will we be at liberty to retrace our steps through the operator structure to unravel the appropriate ordering arrived at implicitly.

‡ We still have to consider the framework in which we quantize equation (2.1). This has earlier been discussed by Pais and Uhlenbeck (1950) and more recently by Kiskis (1974), though the latter was only for the zero-mass situation. If the part of the action A quadratic in the fields Φ is $\Phi P \Phi$ then the propagator for perturbation calculations is thus the inverse of P, with the usual Feynman $i\epsilon$ prescription. This agrees with the results of Taylor (1973, unpublished) in the zero-mass case. The particle interpretation of the quantized field theory can be extended from the zero-mass case of Taylor to the more general nonzero mass situation along the lines of Van Dam and Veltman (1970), and Boulware and Deser (1972) though we will not go into that here.

These have the multiplication table contained in table 1. Since this table is closed it is thus possible to write the general expression

$$Q = \sum_{i=1}^{6} A_i T_i \tag{2.4}$$

and obtain a set of six equations from equation (2.3) and the form (2.4). This set of equations is

$$\begin{aligned}
&(b\Box^2 - \tfrac{1}{4}\Box)A_2 = 1 \\
&2bA_2 = -\tfrac{1}{4}A_6 \\
&[\tfrac{1}{4}\Box + (3a+4b)\Box^2]A_1 + \tfrac{1}{8}\Box^2 A_4 + [(a+b)\Box^2 + \tfrac{1}{8}\Box]A_2 = 0 \\
&[(3a+4b)\Box^2 + \tfrac{1}{4}\Box]A_3 + \tfrac{1}{8}\Box^2 A_5 + \tfrac{1}{8}\Box A_6 - (a+b)\Box A_2 = 0 \\
&[-(3a+4b)\Box]A_1 + (-\tfrac{1}{4}\Box)A_4 - (a+b)\Box A_2 = 0 \\
&-\tfrac{1}{4}\Box A_6 - 2b\Box A_2 = 0.
\end{aligned} \tag{2.5}$$

Table 1. Multiplication table of the tensors $T_1, \ldots T_6$.

	T_1	T_2	T_3	T_4	T_5	T_6
T_1	$4T_1$	T_1	$4T_3$	$\Box T_1$	$\Box T_3$	T_3
T_2	T_1	T_2	T_3	T_4	T_5	T_6
T_3	$\Box T_1$	T_3	$\Box T_3$	$\Box^2 T_1$	$\Box^2 T_3$	$\Box T_3$
T_4	$4T_4$	T_4	$4T_5$	$\Box T_4$	$\Box T_5$	T_5
T_5	$\Box T_4$	T_5	$\Box T_5$	$\Box^2 T_4$	$\Box^2 T_5$	$\Box T_5$
T_6	T_4	T_6	T_5	$\Box T_4$	$\Box T_5$	$\frac{1}{2}\Box T_6 + \frac{1}{2}T_5$

We may solve equations (2.5) to obtain

$$\begin{aligned}
A_1 &= -\left(\frac{\frac{1}{4} + (a+b)\Box}{\frac{1}{2} + (3a+4b)\Box}\right) A_2 \\
A_3 = A_4 &= \left(\frac{(a+2b)}{\frac{1}{2} + (3a+4b)\Box}\right) A_2 \\
A_5 &= \left(\frac{2(a+2b)}{\frac{1}{2} + (3a+4b)\Box}\right) \frac{A_2}{\Box} \\
A_6 &= -8bA_2 \\
A_2 &= [\Box(b\Box - \tfrac{1}{4})]^{-1}.
\end{aligned} \tag{2.6}$$

From equations (2.6) we see that there are three poles in Q which, in terms of the Fourier transform variable p with $\Box = -p^2$, are at

$$p^2 = 0,\ -1/4b,\ 1/(6a+8b). \tag{2.7}$$

The first of these is the physical graviton and the other two may be non-physical. We see from the coefficient A_2 in equations (2.6) that there are only two poles in this term, with opposite coefficients. Since the pole at $p^2 = 0$ is physical, that at $p^2 = -1/4b$ must be unphysical due to its residue with the wrong sign. We cannot say in general if the pole at $p^2 = 1/(6a+8b)$ will also be unphysical since both its value and residue may be physical for suitable choice of the constants a and b. This is a special case of the results of Pais and Uhlenbeck (1950) that ghosts and physical particles interlace each other.

We can investigate the situation more fully by evaluating the residues of Q at the three poles. These residues are given in table 2, where we note in addition that the

Table 2. Residues of the coefficients $A_1, \ldots A_6$ in the complete graviton propagator Q at the three poles it possesses in momentum space.

Pole position in p^2 \ Appropriate term	A_1	A_2	A_3	A_4	A_5	A_6
0	4	-4	$-8(a+2b)$	$-8(a+2b)$	$16(a+2b)/p^2$	$32b$
$-1/4b$	$-\frac{4}{3}$	4	$\frac{16}{3}b$	$\frac{16}{3}b$	$-\frac{128}{3}b^2$	$32b$
$1/(6a+8b)$	$-\frac{2}{3}$	0	$\frac{8}{3}(3a+4b)$	$\frac{8}{3}(3a+4b)$	$4(3a+4b)^3$	0

term A_5 actually has a double pole. The form of the graviton propagators near each of these poles is thus:

(i) near $p^2 = 0$

$$Q_{\alpha\beta,\mu\nu} \sim \frac{2}{p^2}\bigg((\eta_{\alpha\nu}\eta_{\beta\mu}+\eta_{\alpha\mu}\eta_{\beta\nu}-\eta_{\alpha\beta}\eta_{\mu\nu})-\frac{8(a+2b)}{p^2}(\eta_{\alpha\beta}p_\mu p_\nu+\eta_{\mu\nu}p_\alpha p_\beta)$$

$$-\frac{16(a+2b)}{(p^2)^2}p_\alpha p_\beta p_\mu p_\nu+\frac{8b}{p^2}(\eta_{\alpha\mu}p_\beta p_\nu+\eta_{\alpha\nu}p_\beta p_\mu+\eta_{\beta\mu}p_\alpha p_\nu+\eta_{\beta\nu}p_\alpha p_\mu)\bigg) \tag{2.8}$$

(ii) near $p^2 = -1/4b$

$$Q_{\alpha\beta,\mu\nu} \sim \frac{2}{p^2+\frac{1}{4}b}[(\tfrac{2}{3}\eta_{\alpha\beta}\eta_{\mu\nu}-\eta_{\alpha\mu}\eta_{\beta\nu}-\eta_{\alpha\nu}\eta_{\beta\mu})+\tfrac{8}{3}b(\eta_{\alpha\beta}p_\mu p_\nu+\eta_{\mu\nu}p_\alpha p_\beta)$$

$$-\tfrac{64}{3}b^2p_\alpha p_\beta p_\mu p_\nu-4b(\eta_{\alpha\mu}p_\beta p_\nu+\eta_{\alpha\nu}p_\beta p_\mu+\eta_{\beta\mu}p_\alpha p_\nu+\eta_{\beta\nu}p_\alpha p_\mu)] \tag{2.9}$$

(iii) near $p^2 = 1/(6a+8b)$

$$Q_{\alpha\beta,\mu\nu} \sim \frac{1}{p^2-[1/(6a+8b)]}[\tfrac{2}{3}\eta_{\alpha\beta}\eta_{\mu\nu}+\tfrac{8}{3}(3a+4b)(\eta_{\alpha\beta}p_\mu p_\nu+\eta_{\mu\nu}p_\alpha p_\beta)$$

$$-4(3a+4b)^2(p_\alpha p_\beta p_\mu p_\nu)]. \tag{2.10}$$

We can rewrite the term in square brackets in (2.9) in the more recognizable form

$$\tfrac{2}{3}\Theta_{\alpha\beta}\Theta_{\mu\nu}-\Theta_{\alpha\mu}\Theta_{\beta\nu}-\Theta_{\alpha\nu}\Theta_{\beta\mu} \tag{2.11}$$

where $\Theta_{\mu\nu} = \eta_{\mu\nu}+4bp_\mu p_\nu$. The expression (2.11) is the negative of the projection operator for a particle of spin 2 and (mass)^2 of $-1/4b$. The residue of Q at $p^2 = -1/4b$ thus has direct physical significance as massive spin 2 coupling (van Dam and Veltman 1970, Boulware and Deser 1972), though possibly for a tachyon-like particle if b is positive. Since the residue has the opposite sign to that at the massless pole then the particle at $p^2 = -1/4b$ is ghost-like even if it has a real mass, when b is negative. It must thus be eliminated.

The particle at the pole $p^2 = 1/(6a+8b)$ has a real mass if

$$3a+4b > 0. \tag{2.12}$$

Under that condition and the requirement that the graviton is only coupled to a conserved matter stress–energy tensor $T_{\mu\nu}$, (since otherwise inconsistencies would arise in the coupling to the zero-mass pole (Weinberg 1964)) we see that the residue at this third pole has the correct sign and corresponds to a massive scalar particle coupled to the trace of $T_{\mu\nu}$. Thus there is no good physical reason for requiring this pole to be absent if inequality (2.12) can be satisfied. We will turn, then, to the question of cancellation of the residue in Q at $p^2 = -1/4b$, to find if any conditions on a and b allow this to be achieved.

3. Ghost killing for gravity with scalar mesons

We consider first the self-interacting scalar meson Lagrangian

$$L = \sqrt{(-g)}(R + aR^2 + bR_{\mu\nu}R^{\mu\nu} + \tfrac{1}{2}\partial_\mu\Phi\partial_\nu\Phi g^{\mu\nu} - \tfrac{1}{2}m^2\Phi^2 - \tfrac{1}{4}\lambda\phi^4). \tag{3.1}$$

We will discuss the ghost-graviton contributions to lowest order in λ and the gravitational coupling constant; if these cannot be made to cancel then it is not expected that higher-order contributions will help except by some miracle. The relevant lowest-order vertices from equation (3.1) are obtained by making the substitution of $g_{\mu\nu}$ by $\eta_{\mu\nu}+h_{\mu\nu}$, as discussed at the beginning of § 2, to give

$$L = (1+\tfrac{1}{2}h^\alpha_\alpha + \ldots)[(\tfrac{1}{2}\partial_\mu\Phi\partial_\nu\Phi(\eta^{\mu\nu} - h^{\mu\nu}) - \tfrac{1}{2}m^2\Phi^2 - \tfrac{1}{4}\lambda\Phi^4]. \tag{3.2}$$

Thus the lowest-order graviton–four-meson vertex of figure 1 is proportional to

$$\lambda\delta_{\alpha\beta} \tag{3.3}$$

and the graviton–two-meson vertex of figure 2 is proportional to

$$\tfrac{1}{2}\delta_{\alpha\beta}[-m^2-(k_1k_2)] + \tfrac{1}{2}k_{1\alpha}k_{2\beta} \tag{3.4}$$

where k_1 and k_2 are the momenta of the two mesons and (α, β) is the tensor label of the graviton vertex.

Let us consider the process of two-meson scattering with single-graviton exchange as described by figure 3. In lowest order only (3.4) will contribute, with value proportional to

$$\{k_{1\alpha}k_{2\beta} - \delta_{\alpha\beta}[m^2+(k_1k_2)]\}Q^{\alpha\beta\mu\nu}[k^1_{1\mu}k^1_{2\nu} - \delta_{\mu\nu}(m+k^1_1k^1_2)] \tag{3.5}$$

where

$$k_1+k_2 = k^1_1+k^1_2 = p.$$

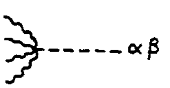

Figure 1. Four-meson–one-graviton vertex, where broken lines represent gravitons, wavy lines represent mesons.

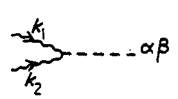

Figure 2. Two-meson–one-graviton vertex.

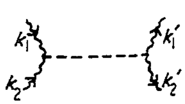

Figure 3. Two-meson scattering with single-graviton exchange.

We take

$$k_1 = (E, \boldsymbol{k}), \qquad k_2 = (E, -\boldsymbol{k}), \qquad k_1^1 = (E, \boldsymbol{k}^1), \qquad (E, \boldsymbol{k}^1) = k_2^1$$

where $|\boldsymbol{k}| = |\boldsymbol{k}^1|$ and $E = (k^2+m^2)^{1/2}$. It is easy to see that the only term in (3.5) involving $(\boldsymbol{k}, \boldsymbol{k}^1)^2$ arises from $k_{1\alpha}k_{2\beta}Q^{\alpha\beta\mu\nu}k^1_{1\mu}k^1_{2\nu}$ with value in terms of the expansion (2.4) of

$$A_1(k_1k_2)(k_1^1k_2^1)+\tfrac{1}{2}A_2[(k_1k_1^1)(k_2k_2^1)+(k_1k_2^1)(k_1^1k_2)]-A_3(k_1k_2)(k_1^1p)(k_2^1p)$$
$$-A_4(k_1^1k_2^1)(k_1p)(k_2p)+A_5(k_1p)(k_2p)(k_1^1p)(k_2^1p)-\tfrac{1}{4}A_6[(k_1k_1^1)(k_2p)(k_1^1p)$$
$$+(k_1k_2^1)(k_1^1p)(k_2p)+(k_1^1k_2)(k_1p)(k_2^1p)+(k_2k_2^1)(k_1p)(k_2p)]. \tag{3.6}$$

The only term in (3.6) depending on $(\boldsymbol{k}\boldsymbol{k}^1)^2$ is that involving A_2, and is equal to

$$\tfrac{1}{2}A_2[(k^2+m^2-\boldsymbol{k}\,.\,\boldsymbol{k}^1)^2+(k^2+m^2+\boldsymbol{k}\,.\,\boldsymbol{k}^1)^2] = A_2[(k^2+m^2)^2+(\boldsymbol{k}\,.\,\boldsymbol{k}^1)^2]. \tag{3.7}$$

Since the term $(\boldsymbol{k}\,.\,\boldsymbol{k}^1)^2$ is independent of the other term in equation (3.7) then it clearly is not possible to remove the pole at $p^2 = -1/4b$ in A_2 by any possible choice of a and b. In detail if we take

$$\boldsymbol{k}\,.\,\boldsymbol{k}^1 = k^2\cos\Theta$$

then the coefficient of $\cos^2\Theta$ in equation (3.6) is given by the last term of equation (3.7), and is equal to

$$\frac{k^2}{p^2(p^2+\frac{1}{4}b)} \tag{3.8}$$

where $p^2 = 4(k^2+m^2)$. Thus at the ghost pole position $p^2 = -1/4b$ the residue in (3.8) of the ghost pole is

$$(\tfrac{1}{4}+4bm^2)$$

and this can be made to vanish only if

$$b = -(1/16m^2). \tag{3.9}$$

The remaining residue at the pole can also be found to vanish identically if account is taken that equation (3.9) corresponds to the condition

$$k^2 = 0. \tag{3.10}$$

For then we obtain

$$k_{1\alpha}k_{2\beta}Q^{\alpha\beta,\mu\nu}k^1_{1\mu}k^1_{2\nu} = m^4(A_1+A_2-8m^2A_3+16m^4A_5-4m^2A_6)$$

which combined with the residues in table 2 gives

$$m^4(-\tfrac{4}{3}+4+\tfrac{8}{3}+\tfrac{8}{3}-8) = 0.$$

The residues at $p^2 = -1/4b$ arising from the remaining terms in (3.5) are also zero since they have values

$$Q_{\alpha\alpha,\mu\mu} = 16A_1+4A_2-8p^2A_3+(p^2)^2A_5-p^2A_6 = -\tfrac{64}{3}+16+\tfrac{32}{3}+\tfrac{8}{3}-8 = 0$$

$$Q_{\alpha\alpha,\mu\nu}k^1_{1\mu}k^1_{2\nu} = m^2\left(4A_1+A_2+\frac{20A_3}{16b}+\frac{A_5}{16b^2}+\frac{A_6}{4b}\right) = m^2(-\tfrac{16}{3}+4+\tfrac{20}{3}+\tfrac{8}{3}-8) = 0$$

The residues at $p^2 = 1/4b$ arising from the remaining terms in (3.5) are also zero, as follows immediately from equation (2.9). Similarly the contribution from the Feynman diagram arising by single-graviton exchange as shown in figure 4 has zero residue, though now the pole is unphysical at $\cos\Theta = (1+2m^2/k^2)$, outside the physical region. The same situation occurs for the diagram of figure 5.

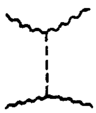

Figure 4. Two-meson scattering with single-graviton exchange

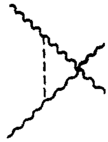

Figure 5. Two-meson scattering with single-graviton exchange.

If we turn to the effect of the interaction $\lambda\Phi^4$ giving the vertex (3.3), we find that the diagrams of figures 6 and 7 are now appropriate. The first is proportional to $Q_{\alpha\alpha,\mu\nu}k_{1\mu}k_{2\nu}$ and the second to $Q_{\alpha\alpha,\beta\beta}$. These both have zero residues at $p^2 = -1/4b$ under the choice (3.9), as we showed in the previous paragraph. Similar cancellation will occur for any higher-power interaction $\sqrt{(-g)}a_n\Phi^n$ added to (3.1). If $n > 4$ this can only produce a coupling to a single graviton which in lowest order is proportional to (3.3), and hence has zero residue if equation (3.9) is satisfied. Since this gives no restriction on the constant a we can satisfy inequality (2.12) if in addition

$$a > \frac{1}{12m^2}. \tag{3.11}$$

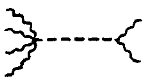

Figure 6. Meson–meson inelastic scattering through single-graviton exchange.

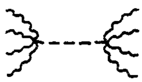

Figure 7. Four-meson scattering with single-graviton exchange.

We conclude that there are no ghost difficulties from equation (3.7) in lowest-order (tree) graphs if (3.9) and (3.11) are both satisfied.

In order to further test the possibility of ghost killing we consider the second-order terms in the expansion of the Lagrangian (3.1) in powers of the quantized graviton field $h_{\mu\nu}$. We will choose a modification of the linear expansion, now taking the quantized field variable $H^{\mu\nu}$ to be defined by

$$g^{\mu\nu}\sqrt{-g} = \eta^{\mu\nu} + \kappa H^{\mu\nu} \tag{3.12}$$

so that in order h^2

$$\sqrt{-g} = 1 + \tfrac{1}{2}\,\mathrm{Tr}\,H + \tfrac{1}{4}[\tfrac{1}{2}(\mathrm{Tr}\,H)^2 - \mathrm{Tr}\,H^2]. \tag{3.13}$$

If we use equations (3.12) and (3.13) in (3.1), the interaction term with two mesons and two gravitons is solely

$$-\tfrac{1}{8}m^2\Phi^2[\tfrac{1}{2}(\mathrm{Tr}\,H)^2 - \mathrm{Tr}\,H^2]. \tag{3.14}$$

This leads to the contribution to the two-meson plus one-graviton scattering process of figure 8, which has solely a single-graviton intermediate state, proportional to

$$e^{\mu\nu}(k_3)Q_{\mu\nu\alpha\beta}e^{\alpha\beta}(k_3^1) \tag{3.15}$$

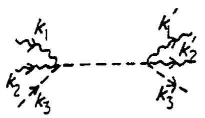

Figure 8. Two-meson–one-graviton scattering with single-graviton exchange.

where k, k^1 are the initial and final graviton momenta with polarization tensors $e^{\mu\nu}(k_3)$ and $e^{\alpha\beta}(k_3^1)$ respectively. We choose for these $e^{\mu}(k_3, \lambda)\, e^{\nu}(k_3, \lambda)$ and $e^{\alpha}(k_3^1, \sigma)\, e^{\beta}(k_3^1, \sigma)$ respectively, where λ and σ take the values 1, 2 corresponding to the two possible helicity states of the gravitons. We take these vectors to be

$$e(k_3, 1) = (0, \boldsymbol{e}_1), \qquad e(k_3, 2) = (0, \boldsymbol{e}_2) \qquad \text{where } \boldsymbol{e}_1 \,.\, \boldsymbol{k}_3 = \boldsymbol{e}_2 \,.\, \boldsymbol{k}_3 = 0.$$

Choosing $\lambda = 1$, $\sigma = 2$ we thus need to evaluate from (3.5) the quantity Q_{1122}, and near the ghost pole at $p^2 = -1/4b$ this takes the value, with $p = (m, \mathbf{0})$, of

$$\frac{2}{(p^2 + \frac{1}{4}b)}[\tfrac{2}{3} - 2(\boldsymbol{e}_1\boldsymbol{e}_2)^2] + \text{non-singular contribution.} \tag{3.16}$$

Thus the residue at the pole in (3.16) is nonzero, so it cannot be cancelled by any further conditions on a and b. Thus the graph of figure 8 has a ghost contribution which cannot be removed internally. Nor will it be cancelled by the contributions from the graphs of figure 9. This can be seen from the fact that all the graphs of figure 9 have at least one vertex of one graviton and two mesons. What is more, if all external particles are on their mass shell then at the ghost pole the two mesons at the vertex under consideration will also be on-shell. This is immediate, since the ghost pole is at $p^2 = 4m^2$, if the mesons are external, and also if one of the mesons is external, since it is associated in all the cases under consideration with on-mass-shell mesons. Thus for the vertices in the second graph of figure 9 we have

$$k_1^1 = k_4^1 = (m, \mathbf{0}), \qquad k_3^1 = (0, \mathbf{0}) \tag{3.17}$$

so

$$k_2^1 = k_3^1 + k_4^1 = (m, \mathbf{0}).$$

Each vertex attached to the internal graviton propagator next to it has then the value

$$Q_{\mu\nu\alpha\beta}(\tfrac{1}{2}m^2\eta_{\alpha\beta} + k_{1\alpha}^1 k_{2\beta}^1).$$

Since near the pole in p^2 at $4m^2$, we have from (2.9) and (3.17) that

$$Q_{\mu\nu\alpha\beta} \sim \frac{2}{(p^2 - 4m^2)}[\tfrac{8}{3}\eta_{\mu\nu} - 2\eta_{\mu\nu} + \tfrac{8}{3}b(4p_\mu p_\nu + p^2\eta_{\mu\nu}) - \tfrac{64}{3}b^2p^2p_\mu p_\nu - 16bp_\mu p_\nu] = 0$$

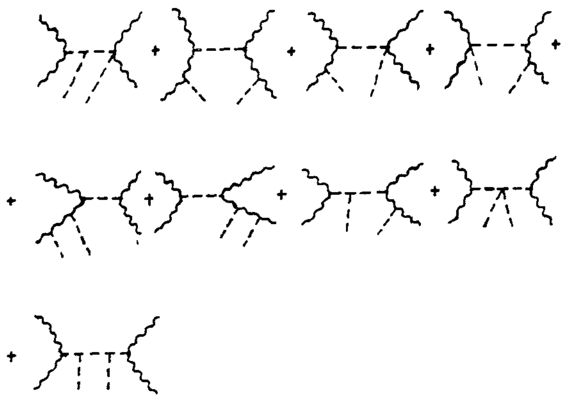

Figure 9. Two external graviton corrections to two-meson scattering through single-graviton exchange.

and

$$Q_{\mu\nu\alpha\beta}k^1_{1\alpha}k^1_{2\beta} \sim \frac{2}{(p^2-4m^2)}\bigg(\tfrac{2}{3}(k^1_1k^1_2)\eta_{\mu\nu}-2k^1_{1\mu}k^1_{2\nu}-\frac{1}{6m^2}(k^1_1k^1_2)p_\mu p_\nu-\frac{1}{6m^2}(k^1_1p)(k^1_2p)\eta_{\mu\nu}$$

$$-\frac{1}{12m^4}(pk^1_1)(pk^1_2)p_\mu p_\nu+\frac{1}{2m^2}(k^1_1p)p_{(\alpha}k^1_{2\beta)}+(k^1_2p)p_{(\alpha}k^1_{1\beta)}\bigg) = 0.$$

Thus each vertex has a zero which is quadratic in the momenta of the external particles as they go to zero at the ghost pole. What is more the vanishing meson denominators in the 2nd, 3rd, 4th, 5th, 6th and 7th graphs of figure 9 are also compensated for quadratically in the momenta by each vertex; for example in the 2nd graph of figure 9 there are two denominators which each introduce an inverse square power of momenta but there are four vertices which completely cancel this effect to give an overall zero contribution. For this reason it is clear that all the graphs of figure 9 give no contribution on-mass-shell to the 2-meson–1-graviton S matrix element. Only figure 8 is left.

4. Ghost killing for scalar mesons

Earlier work ('t Hooft and Veltman 1974, Capper and Duff 1970, Capper *et al* 1974, Deser and van Nieuwenhuizen 1974a, b, De Witt 1964, 1967, Nouri-Moghadam and Taylor 1975a, b) on the quantization of Einstein's gravitational theory has shown that it is necessary to consider further terms in addition to those in (3.1). In particular the additional contribution

$$(D_\mu D^\mu\Phi)^2 \tag{4.1}$$

arose as a counter term necessary to remove certain divergences ('t Hooft and Veltman 1974, Nouri-Moghadam and Taylor 1975a, b). This term will produce ghost mesons and so it is necessary to extend the discussion of the previous sections to this situation. If we consider purely the graph of figure 10 then we can dispense with the gravitational field altogether, so we are left with the Lagrangian

$$L = \tfrac{1}{2}c(\partial_\mu\partial^\mu\Phi)^2+\tfrac{1}{2}(\partial_\mu\Phi)^2-\tfrac{1}{2}m^2\Phi^2 \tag{4.2}$$

for some real constant c. The meson propagator is thus

$$[c(p^2)^2+p^2-m^2]^{-1}$$

which can be written in partial fractions as

$$(\alpha_+-\alpha_-)^{-1}[(p^2-\alpha_+)^{-1}-(p^2-\alpha_-)^{-1}]^{-1} \qquad (4.3)$$

where $\alpha_\pm$ are the roots of

$$cx^2+x-m^2=0$$

so have values

$$\alpha_\pm=\frac{1}{2c}[-1\pm(1+4m^2c)^{1/2}]. \qquad (4.4)$$

We note that the root α_+ is the one which takes the value m^2 as $c\to 0$. It is clear from (4.3) that the pole at $p^2=\alpha_-$ corresponds to a ghost particle with the wrong sign of the residue as compared with the pole at $p^2=\alpha_+$. This former pole is also tachyon-like for any value of c, so it clearly has to be cancelled. However the contribution from figure 10 gives no chance of such cancellation since the residue at the pole is a constant proportional to $(\alpha_+-\alpha_-)^{-1}$ to within factors such as 2π, etc. There is no additional factor of k^2 to cancel the pole by suitable choice of c. Thus the ghost pole is definitely present in lowest order. We conclude that the Lagrangian of equation (4.2) is physically unsatisfactory.

Let us finally turn to the question as to whether or not the term (4.1) is forced on us when starting from the Lagrangian (3.1). This can be estimated by power counting, using the complete form of the graviton propagator (2.4) with its $(p^2)^{-1}$ behaviour at large p^2. The counter term itself arose originally ('t Hooft and Veltman 1974) from loops such as that in figure 11 where the internal propagators for meson and graviton each have asymptotic behaviour $(p^2)^{-1}$, and the two derivatives at each vertex were allowed to act on the external field Φ on each external line. When the graviton propagator behaves asymptotically as $(p^2)^{-2}$ such a term is no longer divergent, so that its counter term is not required.

Figure 10. Three-meson scattering through single-meson exchange.

Figure 11. Single-graviton self-energy correction to meson propagator.

The graviton propagator given by equations (2.4) and (2.6) appears only to behave for large p^2 as $(p^2)^{-1}$, due to the term A_6T_6. However this term does not contribute due to gauge invariance. The effective vertices arising at each end of the graviton propagator will be transverse, and so give no contribution (this can be seen in a particular gauge, when the graviton propagator couples directly to the conserved energy–momentum tensor). Thus the $(p^2)^{-2}$ behaviour of the graviton propagator is justified. Thus the Lagrangian (3.1) is expected to be closed under renormalization effects. We note finally that it is not possible to add further polynomial interactions to (3.1) for renormalizability of the theory, so that as far as the scalar meson is concerned the restrictions on the range of its interactions from renormalizability are identical to those occurring without gravity being present.

5. Conclusions

We have analysed the quantized field theory of scalar mesons interacting with gravitons through a non-minimal Lagrangian in the gravitational field suggested by the recent attempts to renormalize Einstein's Lagrangian. The ghost in the graviton propagator has been found to be present in a certain lower-order graph which cannot be cancelled by any others of the same order. The ghost contribution corresponds to a pole at threshold energy. If it were proposed to cancel this ghost contribution by higher-order graphs, this could only be achieved if the gravitational coupling constant took a particular value. This may indeed be the case, but it would seem a very difficult mechanism to expect to occur, and certainly difficult to investigate further.

If such a ghost killing mechanism does work we have shown that the usual renormalizability criteria for the meson self-interaction have to be reserved unless additional ghost killing occurs for a meson ghost arising from higher-order counter terms.

Our general conclusion is that the Einstein scalar meson Lagrangian modified to include non-minimal counter terms at the single-loop level is physically unsatisfactory because of ghost contributions. We will have to consider other gravitational interactions, especially of photons and fermions, before we can finally conclude that quantization of the Einstein matter Lagrangian is impossible. We will discuss that elsewhere.

Acknowledgments

We would like to thank Dr C J Isham for a helpful discussion associated with the higher-order terms in § 3.

References

Boulware D G and Deser S 1972 *Phys. Rev.* D **6** 3368–82
Capper D M and Duff M J 1974 *Nucl. Phys.* B **62** 147–54
Capper D M and Liebbrandt G 1973 *Nuovo Cim.* **6** 117–9
Capper D M, Duff M J and Halpern L 1974 *Phys. Rev.* D **10** 461
Deser S 1970 *Gen. Rel. Gravitation* **1** 9–18
Deser S and van Nieuwenhuizen P 1974a *Phys. Rev.* D **10** 401–11
——1974b *Phys. Rev.* D **10** 411–20
De Witt B S 1964 *Relativity Groups and Topology* (London: Gordon and Breach)
—— 1967 *Phys. Rev.* **162** 1195, 1239
Feynman R P 1963 *Acta Phys. Pol.* **24** 697
Gupta S 1968 *Phys. Rev.* **172** 1303
't Hooft G and Veltman M 1974 *Ann. Inst. Henri Poincare* **20** 69–94
Keck B W 1971 *PhD Thesis* London University
Kiskis J E Jr 1974 *SLAC-PUB*-1477
Nouri-Moghadam M and Taylor J G 1975a *Proc. R. Soc.* A **344** 87–99
—— 1975b *Kings College Preprint*
—— 1975c *J. Phys.* A*: Math. Gen.* **8** 334–46

Pais A and Uhlenbeck G E 1950 *Phys. Rev.* **79** 148–64
Rivers R J 1964 *Nuovo Cim.* **34** 387
Taylor J G 1974 *J. Phys.* A*: Math., Nucl. Gen.* **7** 12–23
Van Dam H and Veltman M 1970 *Nucl. Phys.* B **22** 397
Weinberg S 1964 *Phys. Rev.* B **134** 882–96